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The Predual and Second Predual of W_α

ZHIJIAN WU*

Department of Mathematics, University of Alabama, Tuscaloosa, Alabama 35487

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We identify the predual and second predual of the space W_α. When $\alpha = 0$, $W_\alpha = BMOA$. Our results are similar to Fefferman's and Sarason's duality theorems. The study of boundedness and compactness of Hankel operators on Dirichlet type spaces is the key to our approach. A new result for the Schatten p-class of this operator is also indicated. © 1993 Academic Press, Inc.

INTRODUCTION

Recall Fefferman's and Sarason's well known theorems (analytic versions)

$$(H^1)^* = BMOA \qquad \text{and} \qquad (VMOA)^* = H^1.$$

The spaces $BMOA$ or $VMOA$ can be also understood as the symbol spaces of analytic functions for which the corresponding Hankel operator on the Hardy space, H^2, is bounded or compact, respectively. These facts suggest that one can study the duality of some spaces in terms of the study of boundedness and compactness of the "related" operators on proper spaces. In this paper, we focus on spaces W_α (which are defined later) and Hankel operators on Dirichlet type spaces.

Let $\mathbf{D}$ be the unit disk of the complex plane $\mathbf{C}$, and $dA = (1/\pi)\, dx\, dy$ be the normalized area measure on $\mathbf{D}$. For $\alpha < 1$, let $dA_\alpha(z) = (2-2\alpha)(1-|z|^2)^{1-2\alpha}\, dA(z)$, and denote by $\langle \cdot, \cdot \rangle_{L^2(dA_\alpha)}$ the inner product in the space $L^2(dA_\alpha)$. The Sobolev space $W^{2,\alpha}$ is a Hilbert space of functions with the inner product

$$\langle u, v \rangle_\alpha = \int_{\mathbf{D}} u(z)\, dA_\alpha \int_{\mathbf{D}} \overline{v(z)}\, dA_\alpha + \langle \partial u, \partial v \rangle_{L^2(dA_\alpha)} + \langle \bar{\partial} u, \bar{\partial} v \rangle^{L^2(dA_\alpha)}.$$

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Here $\partial = \partial/\partial z = \frac{1}{2}(\partial/\partial x - i\,\partial/\partial y)$ and $\bar\partial = \partial/\partial \bar z = \frac{1}{2}(\partial/\partial x + i\,\partial/\partial y)$ denote the usual and the "d-bar" derivatives. The Dirichlet type space D_α ($\alpha < 1$) is the set of those functions in $W^{2,\alpha}$ that are analytic on $\mathbf{D}$. This scale of spaces includes the Bergman space ($\alpha = -\frac{1}{2}$), the Hardy space ($\alpha = 0$), and the classical Dirichlet space ($\alpha = \frac{1}{2}$). (Note that the norms in the first two spaces are different from but equivalent to the corresponding D_α-norms.)

A non-negative measure μ on $\mathbf{D}$ is said to be an α-Carleson measure, see [St], if

$$\int_{\mathbf{D}} |g(z)|^2\, d\mu(z) \leqslant C\, \|g\|_\alpha^2, \qquad \forall g \in D_\alpha.$$

The best constant is denoted by $\|\mu\|_\alpha$. 0-Carleson measures are the classical Carleson measures (see for example [G, p. 238]). For $\alpha \leqslant \frac{1}{2}$, results by Carleson ($\alpha = 0$) and Stegenga [St, Theorems 1.2 and 2.3] say that non-negative measure μ on $\mathbf{D}$ is an α-Carleson measure if and only if

$$\mu[S(I_j)] = O(|I_j|^{1-2\alpha}), \qquad \text{for} \quad \alpha \leqslant 0;$$

$$\mu\left[\bigcup S(I_j)\right] = O\left(\mathrm{Cap}_\alpha\left(\bigcup I_j\right)\right), \qquad \text{for} \quad 0 < \alpha \leqslant \tfrac{1}{2}.$$

holds for any set of intervals $\{I_j\}$ on $\partial\mathbf{D}$ (the boundary of $\mathbf{D}$). Here $S(I)$ is the box based on the interval I defined by $S(I) = \{z : z/|z| \in I, (1 - |I|) \leqslant |z| < 1\}$ and $|I|$ is the normalized length of the interval I on $\partial\mathbf{D}$. Cap_α is the Bessel $(\alpha, 2)$-capacity (its definition can be found in, for example, [Zi, Sect. 2.6]). For $\alpha \leqslant \frac{1}{2}$, there are some other characterizations of the α-Carleson measure and we refer the reader to [KS] and its references for more information. For $\alpha > \frac{1}{2}$, a result in [W2] says that a non-negative measure μ on $\mathbf{D}$ is an α-Carleson measure if and only if $\mu[\mathbf{D}] < \infty$.

For $\alpha < 1$, define W_α as the space of all analytic functions f on $\mathbf{D}$ for which the measure $|f'|^2\, dA_\alpha$ is an α-Carleson measure (see also [RW2]). The W_α norm of f is defined by $\|f\|_{W_\alpha} = \|\, |f'|^2\, dA_\alpha\|_\alpha^{1/2}$. It is a trivial fact that if $\alpha < 0$, W_α is the Bloch space

$$B = \{f \text{ analytic on } \mathbf{D} \colon \sup_{z \in \mathbf{D}} |f'(z)|\,(1 - |z|^2) < \infty\};$$

if $\alpha > \frac{1}{2}$, $W_\alpha = D_\alpha$ (see for example [W2]) and W_0 is $BMOA$ (see for example [G, p. 240]). In general, W_α/C is a Banach space. The atomic decomposition theorems for this scale of spaces can be found in [R] for $\alpha < 0$ and $\alpha > \frac{1}{2}$, [RS] for $\alpha = 0$ and [RW2] for $\alpha \leqslant \frac{1}{2}$.

Preduals and second preduals of B and $BMOA$ are known. Since D_α is a Hilbert space, the predual of D_α is itself. The purpose of this paper is to identify the predual and second predual of W_α for $0 < \alpha \leqslant \frac{1}{2}$. Our method, however, works for the full range of $\alpha < 1$.

We need more notation before we can state our main results.

DEFINITION 1. For $\alpha < 1$, let X_α be the set of all analytic functions f on $\mathbf{D}$ which can be expressed as

$$f = \sum g_j h_j', \qquad g_j, h_j \in D_\alpha.$$

The norm of f in X_α is

$$\|f\|_{X_\alpha} = \inf\left\{\sum \|g_j\|_\alpha \|h_j\|_\alpha : f = \sum g_j h_j'\right\}.$$

The form of this definition is neither strange nor new. In fact, for some spaces, this is one of the natural ways to understand them. For example, a result in [CRW] suggests that the right way to look at a function f in the Hardy space H^1 (on the unit ball) is to write $f = \sum f_j g_j$, where $f_j, g_j \in H^2$, and to understand the H^1 norm of f by $\|f\|_{H^1} = \inf\{\sum \|f_j\|_{H^2} \|g_j\|_{H^2}\}$. We see later that $X_\alpha = A^{1, -2\alpha}$ if $\alpha < 0$; $X_0 = \partial H^1$ and $X_\alpha = \partial D_\alpha = D_{\alpha - 1}$ if $\alpha > \frac{1}{2}$.

DEFINITION 2. For $\alpha < 1$, let w_α be the set of all functions f in W_α for which the little "o" condition,

$$\begin{aligned} \mu[S(I_j)] &= o(|I_j|^{1-2\alpha}), && \text{for} \quad \alpha \leqslant 0, \\ \mu\left[\bigcup S(I_j)\right] &= o\left(\mathrm{Cap}_\alpha\left(\bigcup I_j\right)\right), && \text{for} \quad 0 < \alpha \leqslant \tfrac{1}{2}, \\ \mu[\{r < |z| < 1\}] &= o(1), && \text{for} \quad \alpha \geqslant \tfrac{1}{2}, \end{aligned}$$

holds for any set of intervals $\{I_j\}$ on $\partial\mathbf{D}$. Here μ is the measure given by $|f'|^2 \, dA_\alpha$.

It is easy to show that if $\alpha < 0$, w_α is the little Bloch space

$$B_0 = \{f \in B : f'(z)(1 - |z|^2) \to 0 \text{ as } |z| \to 1_-\};$$

if $\alpha > \frac{1}{2}$, $w_\alpha = D_\alpha$, and $w_0 = VMOA$.

Our main results in this paper are:

THEOREM 1. *For $\alpha < 1$, the dual of X_α is W_α realized by the pairing*

$$\langle f, b \rangle^* = \int_{\mathbf{D}} f(z) \overline{b'(z)} \, dA_\alpha(z), \qquad f \in X_\alpha, \, b \in W_\alpha.$$

THEOREM 2. *For $\alpha < 1$, the dual of w_α is X_α realized by the pairing*

$$*\langle b, f\rangle = \int_{\mathbf{D}} b'(z)\overline{f(z)}\, dA_\alpha(z), \qquad b \in w_\alpha, f \in X_\alpha.$$

For $\alpha = 0$, Theorems 1 and 2 can be proved by using Fefferman's and Sarason's results (we see this later). We include also the known cases $\alpha < 0$ and $\alpha > \frac{1}{2}$ here because our method works uniformly for the range $\alpha < 1$ (although it does not provide a new proof of Fefferman's and Sarason's results).

For convenient reference, we state also the following theorem.

THEOREM 3. *Suppose $p > 1$ and $1/p + 1/q = 1$. Then $\mathscr{H}_b^{(\alpha)} \in S_p$ if and only if $b \in (X_\alpha^q)^*$. More precisely,*

$$\|\mathscr{H}_b^{(\alpha)}\|_{S_p} = \sup\{|\langle f, b\rangle^*| : f \in X_\alpha \text{ and } \|f\|_{X_\alpha^q} \leqslant 1\}.$$

Here $\mathscr{H}_b^{(\alpha)}$ is the Hankel operator defined in the next section. S_p is the Schatten p-class. X_α^q is a generalization of X_α (and X_α is dense in X_α^q!) defined in the last section.

Notations, some known results and several formulas that are needed in this paper are introduced in the next section. Theorems 1 and 2 are proved in Sections 2 and 3, respectively. Inspired by the ideas used in Sections 2 and 3, we discuss the space X_α in a more general setting and prove Theorem 3 in the final section.

1. PRELIMINARIES

Suppose H and K are Hilbert spaces. For $0 < p \leqslant \infty$, the Schatten p-class $S_p = S_p(H, K)$ is the set of all bounded linear operators T from H to K for which the sequence of the singular numbers $\{s_k(T) = \inf\{\|T - R\| : \operatorname{rank}(R) < k\}\}_1^\infty$ belongs to l^p. The S_p norm of T is defined by $\|T\|_{S_p} = \|\{s_k(T)\}_1^\infty\|_{l^p}$. We use $S_0 = S_0(H, K)$ and $S_\infty = S_\infty(H, K)$ for the sets of compact operators and bounded operators from H to K, respectively.

Let T and S be bounded linear operators from H to K and from K to H, respectively. The pairing of T and S is understood by

$$\langle T, S\rangle = \operatorname{trace}(TS).$$

The following standard theorem can be found in, for example, [Zh, Chap. 1].

THEOREM A. *Suppose* $1 \leqslant p < \infty$ *and* $1/p + 1/q = 1$.

(a) $(S_p)^* = S_q$;

(b) $(S_0)^* = S_1$;

(c) (*Schmidt decomposition*) Φ *is a compact operator from* H *to* K *if and only if* Φ *can be written as*

$$\Phi = \sum \lambda_j \langle \cdot, f_j \rangle_H e_j.$$

Here $\{\lambda_j\}_1^\infty$ *is a sequence of numbers tending to* 0, *and* $\{f_j\}_1^\infty$ *and* $\{e_j\}_1^\infty$ *are orthonormal sequences in* H *and* K, *respectively. Moreover, if* $\{\lambda_j\}_1^\infty$ *is in* l^p, *then*

$$\|\Phi\|_{S_p} = \left(\sum |\lambda_j|^p\right)^{1/p}.$$

The point evaluation at any point of $\mathbf{D}$ is a bounded functional on D_α, hence there exists a reproducing kernel for D_α. This kernel has the following expression, see for example [W2],

$$\tilde{K}_\alpha(z, w) = \overline{\tilde{K}_\alpha(w, z)} = 1 + \int_0^z \int_0^{\bar{w}} \frac{1}{(1 - st)^{3 - 2\alpha}} \, ds \, dt.$$

Let P be the set of all analytic polynomials on $\mathbf{D}$. For $\alpha < 1$ and $b \in W^{2,\alpha}$, the (small) Hankel operator with symbol b is defined by (see also [A, AFP, P, R, and W1]):

$$\mathscr{H}_b^{(\alpha)}(g)(w) = \overline{\langle b\bar{g}, K_\alpha(\cdot, w)\rangle_\alpha}, \qquad \forall g \in P.$$

Here $K_\alpha(z, w) = \tilde{K}_\alpha(z, w) - 1$. The definition of $\mathscr{H}_b^{(\alpha)}$ is always fine because $b\bar{g}$ is in $W^{2,\alpha}$ if $b \in W^{2,\alpha}$ and $g \in P$. Instead of K_α, we can also use the kernel $\tilde{K}_\alpha$ in the definition. There is no major difference between the two choices. The one we had implies clearly $\mathscr{H}_b^{(\alpha)}(g)(0) = 0$ and hence is, in practice, easier to work with than the other one.

We always assume our Hankel operator $\mathscr{H}_b^{(\alpha)}$ is continuous in the norm $\|\cdot\|_\alpha$. Hence $\mathscr{H}_b^{(\alpha)}$ can be regarded as an operator on D_α (from D_α to $\overline{D_\alpha}$), because P is dense in D_α. We also use the notation M_f for the operator of multiplication by f, that is $M_f(g)(z) = f(z)\, g(z)$.

The following theorems can be found in [RW1, RW2, W2, and CR (or [R])].

THEOREM B. *Suppose* $\alpha < 1$ *and* b *is analytic on* $\mathbf{D}$. *Then* $\mathscr{H}_b^{(\alpha)}$ *is bounded if and only if* $b \in W_\alpha$. *Moreover,*

$$\|\mathscr{H}_b^{(\alpha)}\| \asymp \|b\|_{W_\alpha}.$$

THEOREM C. *Suppose* $\alpha < 1$ *and* b *is analytic on* $\mathbf{D}$. *Then* $M_{\overline{b'}}: D_\alpha \to L^2(dA_\alpha)$ *is compact if and only if* $b \in w_\alpha$. *Moreover,*

$$\|M_{\overline{b'}}\| \asymp \|b\|_{W_\alpha}.$$

For an analytic function b on $\mathbf{D}$, let $U_b: D_\alpha \to L^2(dA_\alpha)$ be the operator densely defined by

$$U_b(g)(w) = \int_{\mathbf{D}} \frac{\overline{b'(z)}\,(g(w) - g(z))}{(1 - z\bar{w})^{3-2\alpha}}\, dA_\alpha(z), \qquad g \in P.$$

THEOREM D. *Suppose* b *is analytic on* $\mathbf{D}$.

(1) *For* $\alpha < \frac{1}{2}$, U_b *is bounded with* $\|U_b(g)\| \leqslant C\,\|b\|_B\,\|g_\alpha\|$ *if* b *is in* B;

(2) *For* $\alpha = \frac{1}{2}$, U_b *is bounded with* $\|U_b(g)\| \leqslant C\,\|b\|_\alpha\,\|g\|_\alpha$ *if* b *is in* D_α.

Remark 1.1. In [RW2], Theorem D is contained in the proof of Theorem B above for $\alpha \leqslant \frac{1}{2}$.

Suppose $\beta > -1$ and $0 < p < \infty$. The Bergman space $A^{p,\beta}$ is the set of all analytic functions f on $\mathbf{D}$ for which the norm (or semi-norm if $p < 1$)

$$\|f\|_{A^{p,\beta}} = \left\{(1+\beta)\int_{\mathbf{D}} |f(z)|^p\,(1 - |z|^2)^\beta\, dA(z)\right\}^{1/p}$$

is finite.

THEOREM E. *Suppose* $0 < p < \infty$, $\beta > -1$ *and* $\gamma > (1+\beta)/p + \max(1, 1/p)$. *Then* $f \in A^{p,\beta}$ *if and only if there is a sequence* $\{\lambda_j\} \in l^p$ *such that* f *can be expressed as*

$$f(z) = \sum \lambda_j \frac{(1 - |z_j|^2)^{\gamma - (2+\beta)/p}}{(1 - \bar{z}_j z)^\gamma}$$

with

$$\|f\|_{A^{p,\beta}} \asymp \left(\sum |\lambda_j|^p\right)^{1/p}.$$

Here $\{z_j\}$ *is a d-lattice in* $\mathbf{D}$ (*see, for example,* [*R, p.* 230]) *with* $d\,(>0)$ *sufficiently small.*

Remark 1.2. The assumption on γ of Theorem E in [CR, Theorem 2] or [R, Theorem 2.2] is $\gamma > (2+\beta)\max(1, 1/p)$. It is easy to check that we can change to the above assumption (for detail see [W1]). A similar decomposition theorem for the functions in D_α can be obtained by applying the above theorem on the functions in $A^{2,1-2\alpha}$ and then by term by term integration (see also [R, Theorem 2.10]).

$A^{2,\beta}$ is a Hilbert space. It is easy to see that for $\alpha < 1$,

$$\langle g, h\rangle_\alpha = g(0)\,\overline{h(0)} + \langle g', h'\rangle_{A^{2,1-2\alpha}}, \qquad \forall g, h \in D_\alpha. \tag{1}$$

We end this section by setting up three formulas.

For $b \in D_\alpha$ and $g \in P$, by the definition of $\mathscr{H}_b^{(\alpha)}$, we have

$$\mathscr{H}_b^{(\alpha)}(g)(w) = \overline{\langle b\bar{g}, K_\alpha(\cdot, w)\rangle_\alpha} = \overline{\left\langle b'\bar{g}, \frac{\partial}{\partial z} K_\alpha(\cdot, w)\right\rangle_{L^2(dA_\alpha)}},$$

that is

$$\mathscr{H}_b^{(\alpha)}(g)(w) = \left\langle g \frac{\partial}{\partial z} K_\alpha(\cdot, w), b'\right\rangle_{L^2(dA_\alpha)}. \tag{2}$$

For $b \in D_\alpha$ and $f, g \in P$, by formula (2) we have

$$\langle \mathscr{H}_b^{(\alpha)}(g), \bar{f}\rangle_\alpha = \left\langle \frac{\partial}{\partial \bar{w}} \left\langle g \frac{\partial}{\partial z} K_\alpha(\cdot, w), b'\right\rangle_{L^2(dA_\alpha)}, \overline{f'(w)}\right\rangle_{L^2(dA_\alpha(w))}.$$

Denoting $(\partial^2/\partial z\,\partial\bar{w})\,K_\alpha(z, w)$ by $k_\alpha(z, w)$, we can continue the above computation with

$$\begin{aligned} &= \langle g\langle k_\alpha(\cdot, w), \overline{f'(w)}\rangle_{L^2(dA_\alpha(w))}, b'\rangle_{L^2(dA_\alpha)} \\ &= \langle g\langle f'(w), k_\alpha(w, \cdot)\rangle_{L^2(dA_\alpha(w))}, b'\rangle_{L^2(dA_\alpha)}. \end{aligned}$$

Note that

$$k_\alpha(z, w) = \frac{\partial^2}{\partial z\,\partial\bar{w}} K_\alpha(z, w) = \frac{1}{(1 - z\bar{w})^{3-2\alpha}}$$

is in fact the reproducing kernel of $A^{2,1-2\alpha}$ that implies $\langle f', k_\alpha(\cdot, z)\rangle_{L^2(dA_\alpha)} = f'(z)$. Hence we have

$$\langle \mathscr{H}_b^{(\alpha)}(g), \bar{f}\rangle_\alpha = \langle gf', b'\rangle_{L^2(dA_\alpha)}. \tag{3}$$

Remark 1.3. If $b \in W_\alpha$, then both formulas (2) and (3) are valid for $g, h \in D_\alpha$. Furthermore, if $f = \sum g_j h_j'$ with $\sum \|g_j\|_\alpha \|h_j\|_\alpha < \infty$, then by the definition of W_α we can prove easily that

$$\langle f, b\rangle^* = \sum \langle g_j h_j', b\rangle^*, \qquad \forall b \in W_\alpha.$$

Since $\tilde{K}_\alpha(z, w)$ is the reproducing kernel of D_α, we have $\|\tilde{K}_\alpha(\cdot, w)\|_\alpha^2 =$

$\tilde{K}_\alpha(w, w)$. For any $g \in D_\alpha$ and $w \in \mathbf{D}$, the reproducing formula $g(w) = \langle g, \tilde{K}_\alpha(\cdot, w)\rangle_\alpha$ yields the estimate

$$|g(w)| \leqslant (\tilde{K}_\alpha(w, w))^{1/2} \|g\|_\alpha. \tag{4}$$

Similarly, for $g \in A^{2, 1-2\alpha}$ we have

$$|g(w)| \leqslant (k_\alpha(w, w))^{1/2} \|g\|_{A^{2, 1-2\alpha}}. \tag{4'}$$

Remark 1.4. For $\alpha < 1$ and $w \in \mathbf{D}$, we have clearly $k_\alpha(w, w) = (1 - |w|^2)^{2\alpha - 3}$ and the following estimate for $\tilde{K}_\alpha(w, w)$.

$$\tilde{K}_\alpha(w, w) \asymp \begin{cases} (1 - |w|^2)^{2\alpha - 1}, & \text{for} \quad \alpha < \frac{1}{2}; \\ 1 + \log\left(\dfrac{1}{1 - |w|^2}\right), & \text{for} \quad \alpha = \frac{1}{2}; \\ \text{constant depending only on } \alpha, & \text{for} \quad \alpha > \frac{1}{2}. \end{cases}$$

For convenience, we use the number $\|\mathscr{H}_b^{(\alpha)}\|$ to mean the norm $\|b\|_{W_\alpha}$ because of Theorem B.

2. Predual of W_α

Proof of Theorem 1. We first show that $W_\alpha \subseteq (X_\alpha)^*$. Suppose $b \in W_\alpha$, $f \in X_\alpha$, and $f = \sum g_j h_j'$ with $\sum \|g_j\|_\alpha \|h_j\|_\alpha < \infty$. By formula (3) and Remark 1.3,

$$\langle f, b\rangle^* = \sum \langle g_j h_j', b\rangle^* = \sum \langle g_j h_j', b'\rangle_{L^2(dA_\alpha)} = \sum \langle \mathscr{H}_b^{(\alpha)}(g_j), \overline{h_j}\rangle_\alpha$$

and clearly

$$\sum |\langle \mathscr{H}_b^{(\alpha)}(g_j), \bar{h}_j\rangle_\alpha| \leqslant \sum \|\mathscr{H}_b^{(\alpha)}\| \, \|g_j\|_\alpha \, \|h_j\|_\alpha,$$

we get immediately

$$|\langle f, b\rangle^*| \leqslant \|\mathscr{H}_b^{(\alpha)}\| \, \|f\|_{X_\alpha}.$$

We now prove $(X_\alpha)^* \subseteq W_\alpha$. Suppose $T \in (X_\alpha)^*$. For any $g, h \in D_\alpha$, it is clear that $gh' \in X_\alpha$ and $\|gh'\|_{X_\alpha} \leqslant \|g\|_\alpha \|h\|_\alpha$. Hence if in addition $h(0) = 0$, then

$$|T(gh')| \leqslant \|T\| \, \|gh'\|_{X_\alpha} \leqslant \|T\| \, \|g\|_\alpha \, \|h\|_\alpha = \|T\| \, \|g\|_\alpha \, \|h'\|_{A^{2, 1-2\alpha}}.$$

This inequality shows that for fixed $g \in D_\alpha$ the linear map $h' \mapsto T(gh')$ from

$A^{2,1-2\alpha}$ to $\mathbf{C}$ is bounded. Hence by the Riesz–Fischer Theorem, there is a $T_g \in A^{2,1-2\alpha}$ such that

$$T(gh') = \langle h', T_g \rangle_{L^2(dA_\alpha)}.$$

Clearly T_g is uniquely determined by g and the linear map $g \mapsto T_g$ from D_α to $A^{2,1-2\alpha}$ is bounded with $\|T_g\|_{A^{2,1-2\alpha}} \leqslant \|T\| \, \|g\|_\alpha$.

Let $b(z) = b_T(z) = \int_0^z T_1(\zeta)\, d\zeta \in D_\alpha$. For any $g \in D_\alpha$ we have

$$T_g(w) = \langle T_g, k_\alpha(\cdot, w) \rangle_{L^2(dA_\alpha)} = \overline{T(gk_\alpha(\cdot, w))}.$$

Since for fixed $w \in \mathbf{D}$, $gk_\alpha(\cdot, w)$ is always in $A^{2,1-2\alpha}$, we have

$$\begin{aligned} T(gk_\alpha(\cdot, w)) &= \langle gk_\alpha(\cdot, w)\, T_1 \rangle_{L^2(dA_\alpha)} \\ &= \langle gk_\alpha(\cdot, w), b' \rangle_{L^2(dA_\alpha)} \\ &= \frac{\partial}{\partial \bar{w}} \left\langle g \frac{\partial}{\partial z} K_\alpha(\cdot, w), b' \right\rangle_{L^2(dA_\alpha)}. \end{aligned}$$

By formula (2), this implies that for any $g \in P$,

$$T_g(w) = \frac{\partial}{\partial w} \overline{\mathscr{H}_{b_T}^{(\alpha)}(g)(w)}.$$

Thus

$$\|\mathscr{H}_b^{(\alpha)}(g)\|_\alpha = \|T_g\|_{A^{2,1-2\alpha}} \leqslant \|T\| \, \|g\|_\alpha.$$

Hence $\|\mathscr{H}_b^{(\alpha)}\| \leqslant \|T\|$. This is to say, by Theorem B, $b \in W_\alpha$ and then, by Remark 1.3, we have

$$T_g(w) = \frac{\partial}{\partial w} \overline{\mathscr{H}_{b_T}^{(\alpha)}(g)(w)}, \qquad \forall g \in D_\alpha. \tag{5}$$

From the above discussion, we conclude also (since $b \in W_\alpha$)

$$T(gh') = \langle gh', T_1 \rangle_{L^2(dA_\alpha)}, \qquad \forall g, h \in D_\alpha.$$

This implies that the map $T \mapsto b_T$ from $(X_\alpha)^*$ to W_α is bounded and one-to-one. To complete the proof, it remains to verify that

$$T(f) = \langle f, b_T \rangle^*, \qquad \forall f \in X_\alpha.$$

In fact, for any $f = \sum g_j h_j'$ with $\sum \|g_j\|_\alpha \, \|h_j\|_\alpha < \infty$, by formulas (5) and (3) and Remark 1.3,

$$\begin{aligned}T(f) &= \sum T(g_j h_j') = \sum \langle h_j', T_{g_j} \rangle_{L^2(dA_\alpha)} \\ &= \sum \langle h_j', \partial \overline{\mathscr{H}_b^{(\alpha)}(g_j)} \rangle_{L^2(dA_\alpha)} \\ &= \sum \langle \mathscr{H}_b^{(\alpha)}(g_j), \bar{h}_j \rangle_\alpha \\ &= \sum \langle g_j h_j', b' \rangle_{L^2(dA_\alpha)} \\ &= \sum \langle g_j h_j', b_T \rangle^* \\ &= \langle f, b_T \rangle^*.\end{aligned}$$

The proof is therefore complete. ∎

We prove a result about X_α in the rest of this section.

PROPOSITION 1. *Suppose* $\alpha < 1$. *Then*

(a) $X_\alpha = A^{1, -2\alpha}$, *if* $\alpha < 0$;

(b) $X_0 = \partial H^1$;

(c) $X_\alpha = \partial D_\alpha = A^{2, 1-2\alpha}$, *if* $\alpha > \frac{1}{2}$.

Proof. For part (a), assume first $f \in X_\alpha$, $f = \sum g_j h_j'$ and $\sum \|g_j\|_\alpha \|h_j\|_\alpha \leqslant 2 \|f\|_{X_\alpha}$. Estimating the $A^{1, -2\alpha}$ norm of f directly, we get

$$\begin{aligned}\|f\|_{A^{1,-2\alpha}} \leqslant \sum \|g_j h_j'\|_{A^{1,-2\alpha}} &= C \sum \int_{\mathbf{D}} |g_j(z) h_j'(z)| (1-|z|^2)^{-2\alpha} \, dA(z) \\ &\leqslant C \sum \left(\int_{\mathbf{D}} |g_j(z)|^2 (1-|z|^2)^{-2\alpha-1} \, dA(z) \right)^{1/2} \\ &\quad \times \left(\int_{\mathbf{D}} |h_j'(z)|^2 (1-|z|^2)^{1-2\alpha} \, dA(z) \right)^{1/2}.\end{aligned}$$

Since $\alpha < 0$, we have (easy to check)

$$\left(\int_{\mathbf{D}} |g_j(z)|^2 (1-|z|^2)^{-2\alpha-1} \, dA(z) \right)^{1/2} \asymp \|g_j\|_\alpha.$$

By formula (1),

$$\left(\int_{\mathbf{D}} |h_j'(z)^2 (1-|z|^2)^{1-2\alpha} \, dA(z) \right)^{1/2} = C \|h_j'\|_{A^{2,1-2\alpha}} \leqslant C \|h_j\|_\alpha.$$

Hence $\|f\|_{A^{1,-2\alpha}} \leqslant C \sum \|g_j\|_\alpha \|h_j\|_\alpha \leqslant C \|f\|_{X_\alpha}$.

Suppose $f \in A^{1,-2\alpha}$. By Theorem E, f can be expressed as

$$f(z) = \sum \lambda_j \frac{(1-|z_j|^2)^{3-2\alpha}}{(1-\bar{z}_j z)^{5-4\alpha}}$$

with $\|f\|_{A^{1,-2\alpha}} \asymp \sum |\lambda_j|$. Since

$$\frac{(1-|z_j|^2)^{3-2\alpha}}{(1-\bar{z}_j z)^{5-4\alpha}} = \frac{(1-|z_j|^2)^{3/2-\alpha}}{(1-\bar{z}_j z)^{2-2\alpha}} \frac{(1-|z_j|^2)^{3/2-\alpha}}{(1-\bar{z}_j z)^{3-2\alpha}}$$

and straightforward computation shows that the estimation

$$\left\| \frac{(1-|z_j|^2)^{3/2-\alpha}}{(1-\bar{z}_j z)^{2-2\alpha}} \right\|_\alpha, \qquad \left\| \frac{(1-|z_j|^2)^{3/2-\alpha}}{(1-\bar{z}_j z)^{3-2\alpha}} \right\|_{A^{2,1-2\alpha}} \leqslant C$$

is independent of j. Hence

$$\|f\|_{X_\alpha} \leqslant C \sum |\lambda_j| \asymp \|f\|_{A^{1,-2\alpha}}.$$

We now prove part (b). Suppose $f \in H^1$. Then f can be written as $f = gh$ with $g, h \in H^2$ and $\|f\|_{H^1} = \|g\|_{H^2} \|h\|_{H^2}$. Since $\|\cdot\|_0 \asymp \|\cdot\|_{H^2}$ and $f' = gh' + hg'$, we have

$$\|f'\|_{X_0} \leqslant 2 \|g\|_0 \|h\|_0 \asymp \|g\|_{H^2} \|h\|_{H^2} = \|f\|_{H^1}.$$

This implies $\partial H^1 \subseteq X_0$. To prove $X_0 \subseteq \partial H^1$, we only need to show $\forall g, h \in D_0 = H^2$, the function

$$F(z) = \int_0^z g(\zeta)\, h'(\zeta)\, d\zeta$$

is in H^1 and $\|F\|_{H^1} \leqslant C \|g\|_0 \|h\|_0$.

Let $F_r(z) = F(rz)$, $r \in (0, 1)$. By Fefferman's duality theorem, we have

$$\|F_r\|_{H^1} = \sup_{b \in W_0,\ \|b\|_{W_0} \leqslant 1} |\langle F_r, b \rangle_{H^2}|.$$

An easy computation shows

$$\begin{aligned} \langle F_r, b \rangle_{H^2} &= \int_{\mathbf{D}} rF'(rz)\, \overline{b'(z)} \log\left(\frac{1}{|z|^2}\right) dA(z) \\ &= \int_{\mathbf{D}} rg(rz)\, h'(rz)\, \overline{b'(z)} \log\left(\frac{1}{|z|^2}\right) dA(z). \end{aligned}$$

This implies (write $g_r(z)=g(rz)$ and $h_r(z)=h(rz)$)

$$|\langle F_r, b\rangle_{H^2}| \leqslant C\left(\int_{\mathbf{D}} |g(rz)\, b'(z)|^2 (1-|z|^2)\, dA(z)\right)^{1/2}$$
$$\times\left(\int_{\mathbf{D}} |h'(rz)|^2 (1-|z|^2)\, dA(z)\right)^{1/2}$$
$$\leqslant C\, \|g_r\|_0\, \|b\|_{W_0}\, \|h_r\|_0$$
$$\leqslant C\, \|g\|_0\, \|h\|_0.$$

That is $\|F\|_{H^1} \leqslant C\, \|g\|_0\, \|h\|_0$.

Finally we prove part (c). Assume $f\in A^{2,1-2\alpha}$ and $f=F'$ with $F(0)=0$. Then $F\in D_\alpha$ and $\|F\|_\alpha = \|f\|_{A^{2,1-2\alpha}}$. Hence

$$\|f\|_{X_\alpha} = \|1F'\|_{X_\alpha} \leqslant \|1\|_\alpha\, \|F\|_\alpha \leqslant \|f\|_{A^{2,1-2\alpha}}.$$

Suppose $f\in X_\alpha$ with $f=\sum g_j h_j'$ and $\sum \|g_j\|_\alpha\, \|h_j\|_\alpha \leqslant 2\, \|f\|_{X_\alpha}$. By formula (4) and Remark 1.4, we have $|g_j(z)| \leqslant C\, \|g_j\|_\alpha$. Straightforward estimation hence yields

$$\|f\|_{A^{2,1-2\alpha}} \leqslant \sum \|g_j h_j'\|_{A^{2,1-2\alpha}}$$
$$\leqslant \sum (\sup_{z\in\mathbf{D}} |g_j|)\, \|h_j'\|_{A^{2,1-2\alpha}}$$
$$\leqslant C \sum \|g_j\|_\alpha\, \|h_j\|_\alpha$$
$$\leqslant C\, \|f\|_{X_\alpha}. \quad \blacksquare$$

Remark 2.1. Comparing $\langle\cdot,\cdot\rangle^*$ and $^*\langle\cdot,\cdot\rangle$ with the pairing $\langle\cdot,\cdot\rangle_{H^2}$, one sees that in the case of $\alpha=0$, Theorems 1 and 2 are immediate consequences of Fefferman's and Sarason's duality theorems and part (b) of the above proposition.

3. Second Predual of W_α

We first prove the following proposition.

PROPOSITION 2. *For $\alpha<1$ and b analytic on* $\mathbf{D}$, *$\mathscr{H}_b^{(\alpha)}$ is compact from D_α to $\overline{D_\alpha}$ if and only if b is in w_α.*

Remark 3.1 For $\alpha>\frac{1}{2}$, Proposition 2 was proved in [W2]. For $\alpha\leqslant 0$, it can be reduced to results by Peller [P] or Rochberg [R]. The following proof is good for $\alpha\leqslant\frac{1}{2}$.

Proof. The "if" part of the proposition is easy. In fact, by Theorem C, b is in w_α implies the map $M_{\overline{b'}}: D_\alpha \mapsto L^2(dA_\alpha)$ is compact. Since the orthogonal projection from $L^2(dA_\alpha)$ onto $A^{2,1-2\alpha}$ is bounded and has the expression $\langle \cdot, k_\alpha(\cdot, w)\rangle_{L^2(dA_\alpha)}$, we then have that the map from D_α to $\overline{A^{2,1-2\alpha}}$ defined by the formula

$$g(w) \mapsto \overline{\langle \overline{M_{\overline{b'}}(g)}, k_\alpha(\cdot, w)\rangle_{L^2(dA_\alpha)}} = \overline{\langle b'\bar{g}, k_\alpha(\cdot, w)\rangle_{L^2(dA_\alpha)}} = \bar{\partial}\mathscr{H}_b^{(\alpha)}(g)(w)$$

is compact. This is equivalent to $\mathscr{H}_b^{(\alpha)}$ being compact from D_α to $\overline{D_\alpha}$.

Suppose now $\mathscr{H}_b^{(\alpha)}$ is compact; to prove the "only if" part, we need to show that $M_{\overline{b'}}$ is a compact operator from D_α to $L^2(dA_\alpha)$. Note that $M_{\overline{b'}} - \bar{\partial}\mathscr{H}_b^{(\alpha)} = U_b$. We then reduce to show U_b is compact from D_α to $L^2(dA_\alpha)$.

For $r \in (0, 1)$ and $z \in \mathbf{D}$, let $b_r(z) = b(rz)$. It is clear that each of the operators U_{b_r} is compact. (In fact, U_{b_r} is in S_2, because (see [W2, Theorem 2]) both $M_{\overline{F'}}$ and $\bar{\partial}\mathscr{H}_F^{(\alpha)}$ are in $S_2(D_\alpha, L^2(dA_\alpha))$ if the analytic function F satisfies $\int_{\mathbf{D}} |F'(z)|^2 \log(1/(1-|z|^2))\, dA(z) < \infty$.) By Theorem D, we have

$$\|U_b - U_{b_r}\| \leqslant \begin{cases} C\, \|b - b_r\|_B, & \text{if } \alpha < \frac{1}{2} \text{ and } b \in B; \\ C\, \|b - b_r\|_\alpha, & \text{if } \alpha = \frac{1}{2} \text{ and } b \in D_{1/2}. \end{cases}$$

Note that $\|b - b_r\|_B \to 0$ and $\|b - b_r\|_\alpha \to 0$ as $r \to 1_-$, if $b \in B_0$ and $b \in D_\alpha$, respectively. To complete the proof, it remains to show that the compactness of $\mathscr{H}_b^{(\alpha)}$ implies $b \in B_0 \cap D_\alpha$.

In fact $1 \in D_\alpha$ implies $\mathscr{H}_b^{(\alpha)}(1) \in \overline{D_\alpha}$. By formula (2)

$$\mathscr{H}_b^{(\alpha)}(1)(w) = \left\langle \frac{\partial}{\partial z} K_\alpha(\cdot, w), b' \right\rangle_{L^2(dA_\alpha)} = \langle K_\alpha(\cdot, w), b\rangle_\alpha = \overline{b(w) - b(0)},$$

thus $b \in D_\alpha$. On the other hand, let

$$f_w(z) = \frac{(1-|w|^2)^{1/2}}{(1-\bar{w}z)^{1-\alpha}}, \qquad w \in \mathbf{D}.$$

It is easy to check that $\|f_w\|_\alpha \leqslant C$ uniformly for $w \in \mathbf{D}$ and that $f_w \to 0$ weakly in D_α as $|w| \to 1_-$. Hence the fact $f_w(z) f'_w(z) = (1-\alpha)\, \bar{w}(1-|w|^2)\, k_\alpha(z, w)$ and the following computation using formula (3),

$$\langle \mathscr{H}_b^{(\alpha)}(f_w), \overline{f_w}\rangle_\alpha = \langle f_w f'_w, b'\rangle_{L^2(dA_\alpha)} = (1-\alpha)\, \bar{w}(1-|w|^2)\, \overline{b'(w)},$$

together with the estimate

$$(1-|w|^2)\,|b'(w)| \leqslant C\,\|\mathscr{H}_b^{(\alpha)}(f_w)\|_\alpha$$

yield the desired result and thus complete the proof. ∎

Proof of Theorem 2. The part $X_\alpha \subseteq (w_\alpha)^*$ is easy. In fact, suppose $f=\sum g_j h_j' \in X_\alpha$ and $\sum \|g_j\|_\alpha \|h_j\|_\alpha < \infty$. For any $b \in w_\alpha$, by Remark 1.3 we get

$$^*\langle b, f\rangle = \sum {}^*\langle b, g_j h_j'\rangle = \sum \langle b', g_j h_j'\rangle_{L^2(dA_\alpha)} = \sum \langle \overline{h_j}, \mathscr{H}_b^{(\alpha)}(g_j)\rangle_\alpha.$$

Therefore the estimate

$$\sum |\langle \overline{h_j}, \mathscr{H}_b^{(\alpha)}(g_j)\rangle_\alpha| \leqslant \sum \|h_j\|_\alpha \|\mathscr{H}_b^{(\alpha)}(g_j)\|_\alpha \leqslant \|\mathscr{H}_b^{(\alpha)}\|\, \|g_j\|_\alpha \|h_j\|_\alpha$$

yields immediately

$$|^*\langle b, f\rangle| \leqslant \|\mathscr{H}_b^{(\alpha)}\|\, \|f\|_{X_\alpha}.$$

To prove $(w_\alpha)^* \subseteq X_\alpha$, we consider the map $b \mapsto \partial\overline{\mathscr{H}_b^{(\alpha)}}$ from w_α to $S_0(\overline{D_\alpha}, A^{2, 1-2\alpha})$. This map is clearly one-to-one and maps w_α onto a closed subspace of S_0. Let $L \in (w_\alpha)^*$. Extend L to a bounded linear functional $\tilde{L}$ on S_0 such that $\|\tilde{L}\| = \|L\|$. By Theorem A(b), there is a Φ in $S_1(A^{2, 1-2\alpha}, \overline{D_\alpha})$ such that $\|\Phi\|_{S_1} = \|\tilde{L}\|$ and for any $T \in S_0$, $\tilde{L}(T) = \langle T, \Phi\rangle$. Suppose the Schmidt decomposition of Φ is (by Theorem A(c))

$$\Phi = \sum s_j \langle \cdot, f_j\rangle_{L^2(dA_\alpha)}\, \overline{g_j}.$$

Here $\{s_j\}_1^\infty$ are the singular numbers of Φ, and $\{f_j\}_1^\infty$ and $\{g_j\}_1^\infty$ are orthonormal sequences in $A^{2, 1-2\alpha}$ and D_α, respectively.

By formula (1), $\{h_j(z) = \int_0^z f_j(\zeta)\, d\zeta\}_0^\infty$ is clearly an orthonormal sequence in D_α. Let

$$f = f_L = \sum s_j g_j f_j = \sum s_j g_j h_j'. \tag{6}$$

Then clearly $f \in X_\alpha$, and

$$\|f\|_{X_\alpha} \leqslant \sum |s_j| = \|\Phi\|_{S_1} = \|L\|.$$

For any $b \in w_\alpha$, we have

$$
\begin{aligned}
L(b) &= \tilde{L}(\partial \overline{\mathscr{H}_b^{(\alpha)}}) = \langle \partial \overline{\mathscr{H}_b^{(\alpha)}}, \Phi \rangle \\
&= \operatorname{trace}(\partial \overline{\mathscr{H}_b^{(\alpha)}} \Phi) = \operatorname{trace}(\Phi \, \partial \overline{\mathscr{H}_b^{(\alpha)}}) \\
&= \sum \langle \Phi \, \partial \overline{\mathscr{H}_b^{(\alpha)}}(\overline{g_j}), \overline{g_j} \rangle_\alpha \\
&= \sum s_j \langle \partial \overline{\mathscr{H}_b^{(\alpha)}(g_j)}, f_j \rangle_{L^2(dA_\alpha)} \\
&= \sum s_j \langle b', g_j h_j' \rangle_{L^2(dA_\alpha)} \\
&= \langle b', f \rangle_{L^2(dA_\alpha)},
\end{aligned}
$$

thus

$$
L(b) = {}^*\langle b, f \rangle, \qquad \forall b \in w_\alpha. \tag{7}
$$

This implies $\|L\| \leqslant \|f\|_{X_\alpha}$, and hence $\|L\| = \|f\|_{X_\alpha}$.

To complete the proof it remains to show that the map $L \mapsto f_L$ defined by formula (6) is well defined and one-to-one. Identity (7) plays an important role in the following argument.

For any $\zeta \in \mathbf{D}$, let $b_\zeta(z) = (\partial/\partial\bar{\zeta})\, K_\alpha(z, \zeta)$. By formula (2) and the fact that $b_\zeta'(z) = k_\alpha(z, \zeta)$, we get for any $g \in P$

$$
\mathscr{H}_{b_\zeta}^{(\alpha)}(g)(w) = \left\langle g \frac{\partial}{\partial z} K_\alpha(\cdot, w), b_\zeta' \right\rangle_{L^2(dA_\alpha)} = \frac{\partial}{\partial \zeta} K_\alpha(\zeta, w)\, g(\zeta).
$$

This shows that $\mathscr{H}_{b_\zeta}^{(\alpha)}$ is a compact operator (rank one indeed!). Thus $b_\zeta \in w_\alpha$. The "well defined" part follows from the computation using formula (7):

$$
L(b_\zeta) = {}^*\langle b_\zeta, f_L \rangle = \langle b_\zeta', f_L \rangle_{L^2(dA_\alpha)} = \overline{f_L(\zeta)}.
$$

The "one-to-one" part is then a immediate consequence of the identity (7). ∎

Remark 3.2. The above proof also shows that functions $f \in X_\alpha$ can always be represented as

$$
f = \sum s_j g_j h_j' \qquad \text{with} \quad \|f\|_{X_\alpha} = \sum |s_j|,
$$

where $\{g_j\}$ and $\{h_j\}$ are orthonormal sequences in D_α.

4. Other Related Results

A sequence $\{e_j\}$ in a Hilbert space H is called WO (weakly orthonormal) if $\{e_j\}$ is the image of an orthonormal sequence under a bounded linear map on H.

PROPOSITION 3. *Let $\alpha<1$ and $1\leqslant p<\infty$. Suppose $\{g_j\}$ and $\{f_j\}$ are* WO *sequences in D_α and $A^{2,1-2\alpha}$, respectively. Then*

$$f=\sum s_j g_j f_j$$

converges uniformly in $r\mathbf{D}$ for any fixed $r\in(0,1)$ if $\{s_j\}_1^\infty$ belongs to l^p.

Proof. The proposition is clearly true if $p=1$. Suppose $p>1$ and $1/p+1/q=1$. Without loss of generality, assume furthermore that $\{g_j\}$ and $\{f_j\}$ are images of the orthonormal bases $\{e_j\}_1^\infty$ in D_α and $\{\tilde{e}_j\}_1^\infty$ in $A^{2,1-2\alpha}$ under the bounded linear maps Φ and Ψ, respectively. Using formulas (4) and (4′), we have the estimate

$$\begin{aligned}\sum |s_j g_j(z) f_j(z)| &\leqslant \left(\sum |s_j|^p\right)^{1/p}\left(\sum |g_j(z) f_j(z)|^q\right)^{1/q}\\ &\leqslant \|\{s_j\}\|_{l^p}\left(\sum |g_j(z)|^{2q}\right)^{1/2q}\left(\sum |f_j(z)|^{2q}\right)^{1/2q}\\ &\leqslant \|\{s_j\}\|_{l^p}\,(\tilde{K}_\alpha(z,z)\,k_\alpha(z,z))^{(q-1)/2q}\\ &\quad\times\left(\sum |g_j(z)|^2\right)^{1/2q}\left(\sum |f_j(z)|^2\right)^{1/2q}.\end{aligned}$$

Hence we need only show that these last two terms are bounded uniformly on $r\mathbf{D}$.

Let $[\Phi]$ be the matrix with the jkth entry defined by

$$\varphi_{jk}=\langle \Phi(e_j), e_k\rangle_\alpha.$$

$[\Phi]$ is clearly a bounded matrix operator on l^2 and $\|[\Phi]\|=\|\Phi\|$. Regard $\{g_j(z)\}$ and $\{e_j(z)\}$ as sequences in l^2, we have

$$g_j(z)=\Phi(e_j)(z)=\sum_k \varphi_{jk}\, e_k(z),$$

hence

$$\sum_j |g_j(z)|^2=\sum_j \left|\sum_k \varphi_{jk}\, e_k(z)\right|^2 \leqslant \|[\Phi]\|^2 \sum_j |e_j(z)|^2.$$

Since $\{e_j\}_1^\infty$ is an orthonormal base of D_α, we have

$$\tilde{K}_\alpha(z,w)=\sum_j e_j(z)\,\overline{e_j(w)}.$$

This implies $\sum_j |e_j(z)|^2 = \tilde{K}_\alpha(z, z)$. Hence

$$\sum_j |g_j(z)|^2 \leqslant \|\Phi\|^2 \tilde{K}_\alpha(z, z).$$

Similar to the above argument, we can get

$$\sum_j |f_j(z)|^2 \leqslant \|\Psi\|^2 k_\alpha(z, z).$$

The proof is complete. ∎

DEFINITION 3. For $\alpha < 1$ and $1 \leqslant p < \infty$, let X_α^p be the set of all analytic functions f on $\mathbf{D}$ which can be expressed as

$$f = \sum s_j g_j h_j'.$$

Here $\{s_j\}_1^\infty$ is in l^p, $\{g_j\}$ and $\{h_j\}$ are WO sequences in D_α with $\|g_j\|_\alpha$, $\|h_j\|_\alpha \leqslant 1$. The norm of f in X_α^p is defined by

$$\|f\|_{X_\alpha^p} = \inf\left\{\left(\sum |s_j|^p\right)^{1/p} : f = \sum s_j g_j h_j'\right\}.$$

Remark 4.1. X_α^p is clearly a Banach space. It is easy to check that, for $p > 1$, $X_\alpha^1 \subsetneq X_\alpha^p$, X_α^1 is dense in X_α^p, and the inclusion $X_\alpha^1 \subset X_\alpha^p$ is continuous. By Remark 3.2 and the above definition, it is also easy to see that $X_\alpha = X_\alpha^1$. For $p > 1$, the WO condition in the definition is necessary, otherwise the X_α^p norm of gh', for $g, h \in D_\alpha$, will be zero always. In fact, write $gh' = \sum_1^n (1/n)\, gh'$, then

$$\|gh'\|_{X_\alpha^p}^p \leqslant C \inf_n \sum_1^n \left(\frac{1}{n}\right)^p = 0.$$

PROPOSITION 4. *Suppose $1 < p < \infty$ and $\alpha < 1$. $(X_\alpha^p)^*$ can be identified as the set of all analytic functions b on* $\mathbf{D}$ *such that*

$$|\langle f, b\rangle^*| \leqslant C\, \|f\|_{X_\alpha^p}, \qquad \forall f \in X_\alpha. \tag{8}$$

Proof. Suppose $T \in (X_\alpha^p)^*$. Then the continuous inclusion $X_\alpha \subset X_\alpha^p$ implies that the restriction of T to X_α belongs to $(X_\alpha)^* = W_\alpha$. Thus by Theorem 1, there is a unique $b \in W_\alpha$ satisfying

$$T(f) = \langle f, b\rangle^*, \qquad \forall f \in X_\alpha.$$

Together with the definition of X_α^p, this is enough to conclude the desired result. ∎

Proof of Theorem 3. Suppose $1/p+1/q=1$. First we consider that $b\in(X_\alpha^q)^*$. We want to show that $\mathscr{H}_b^{(\alpha)}$ is in S_p. To do this we need only have

$$|\langle T, \mathscr{H}_b^{(\alpha)}\rangle| \leqslant C\,\|T\|_{S_q}, \qquad \forall T\in S_q(\overline{D_\alpha}, D_\alpha).$$

Suppose the Schmidt decomposition for such a $T\in S_q(\overline{D_\alpha}, D_\alpha)$ is

$$T=\sum t_j\langle\cdot, \overline{h_j}\rangle_\alpha\, g_j.$$

Here $\{h_j\}_1^\infty$ and $\{g_j\}_1^\infty$ are orthonormal sequences in D_α. Clearly $\|T\|_{S_q}=(\sum|t_j|^q)^{1/q}$. Computing the pairing of T and $\mathscr{H}_b^{(\alpha)}$, we have

$$\begin{aligned}\langle T, \mathscr{H}_b^{(\alpha)}\rangle &= \operatorname{trace}(T\mathscr{H}_b^{(\alpha)})\\ &=\sum\langle T\mathscr{H}_b^{(\alpha)}(g_j), g_j\rangle_\alpha\\ &=\sum t_j\langle\mathscr{H}_b^{(\alpha)}(g_j), \overline{h_j}\rangle_\alpha\\ &=\sum t_j\langle g_jh_j', b'\rangle_{L^2(dA_\alpha)}.\end{aligned}$$

Define $f=\sum t_jg_jh_j'$ and $f_n=\sum_1^n t_jg_jh_j'$. Clearly $f\in X_\alpha^q$, $f_n\in X_\alpha$, and $\lim_{n\to\infty}\|f_n\|_{X_\alpha^q}=\|f\|_{X_\alpha^q}\leqslant\|T\|_{S_q}$. Hence

$$\begin{aligned}|\langle T, \mathscr{H}_b^{(\alpha)}\rangle| &= \left|\lim_{n\to\infty}\sum_1^n t_j\langle g_jh_j', b'\rangle_{L^2(dA_\alpha)}\right|\\ &=\lim_{n\to\infty}|\langle f_n, b'\rangle_{L^2(dA_\alpha)}|\\ &\leqslant\lim_{n\to\infty}\|b\|_{(X_\alpha^q)^*}\|f_n\|_{X_\alpha^q}\\ &\leqslant\|b\|_{(X_\alpha^q)^*}\|T\|_{S_q}.\end{aligned}$$

We now assume $\mathscr{H}_b^{(\alpha)}\in S_p$. Suppose $f\in X_\alpha$ (it is dense in X_α^q!) and $f=\sum s_jg_jh_j'$ with $\{g_j;\|g_j\|_\alpha\leqslant 1\}$ and $\{h_j;\|h_j\|_\alpha\leqslant 1\}$ being WO in D_α, and $\sum|s_j|^q<\infty$. Without loss of generality, we suppose further that both $\{g_j\}$ and $\{h_j\}$ are the images of the basis $\{e_j\}_1^\infty$ under bounded linear operators Φ and Ψ with $\|\Phi\|, \|\Psi\|\leqslant 1$. Let

$$T=\sum s_j\langle\cdot, \overline{e_j}\rangle_\alpha\, e_j.$$

Clearly T is in S_q with $\|T\|_{S_q}=(\sum|s_j|^q)^{1/q}$. Since $\mathscr{H}_b^{(\alpha)}\in S_p$ implies $b\in W_\alpha$, we have

$$\begin{aligned}\langle f, b\rangle^* &= \sum s_j \langle g_j h_j', b'\rangle_{L^2(dA_\alpha)} \\ &= \sum s_j \langle \mathscr{H}_b^{(\alpha)} \Phi(e_j), \overline{\Psi(e_j)}\rangle_\alpha \\ &= \sum s_j \langle \overline{(\Psi)^*}\, \mathscr{H}_b^{(\alpha)}\, \Phi(e_j), \overline{e_j}\rangle_\alpha \\ &= \langle T, \overline{(\Psi)^*}\, \mathscr{H}_b^{(\alpha)} \Phi\rangle\end{aligned}$$

(here $(\Psi)^*$ is the conjugate of Ψ); we have then

$$|\langle f, b\rangle^*| \leqslant \|T\|_{S_q} \|\Psi\| \|\Phi\| \|\mathscr{H}_b^{(\alpha)}\|_{S_p} \leqslant \left(\sum |s_j|^q\right)^{1/q} \|\mathscr{H}_b^{(\alpha)}\|_{S_p}.$$

This implies

$$|\langle f, b\rangle^*| \leqslant \|f\|_{X_\alpha^q} \|\mathscr{H}_b^{(\alpha)}\|_{S_p}.$$

The proof is complete. ∎

A natural question is how to identify $(X_\alpha^p)^{**}$. We have the following result.

PROPOSITION 5. *Suppose* $1 < p < \infty$. *Then* $(X_\alpha^p)^{**} = X_\alpha^p$ *identified by the following map from* $(X_\alpha^p)^{**}$ *to* X_α^p:

$$L \mapsto f_L(z) = L\left(\frac{\partial}{\partial \bar{z}} K_\alpha(\cdot, z)\right). \tag{9}$$

Proof. We only need to show that $(X_\alpha^p)^{**} \subseteq X_\alpha^p$ and the map (9) is one-to-one. By Proposition 4 and Theorem 3, we can regard $(X_\alpha^p)^*$ as a closed subspace of S_q. By Theorem A(a), we can identify $L \in (X_\alpha^p)^{**}$ by a linear operator in S_p. Using the method in the proof of Theorem 2, we can show that $\|f_L\|_{X_\alpha^p} = \|L\|$ and hence the map (9) is one-to-one. ∎

PROPOSITION 6. *Suppose* $\alpha < 1$, $p \geqslant 1$, *and* $(1-\alpha)p > \frac{1}{2}$. *Then* $A^{p,(2-2\alpha)p-2} \subseteq X_\alpha^p$.

Proof. By Theorem E, a function $f \in A^{p,(2-2\alpha)p-2}((1-\alpha)p > \frac{1}{2})$ can always be expressed as

$$f = \sum \lambda_j \frac{(1-|z_j|^2)^{3-2\alpha}}{(1-\bar{z}_j z)^{5-4\alpha}}$$

with $\|f\|_{A^{p,(2-2\alpha)p-2}} \asymp (\sum |\lambda_j|^p)^{1/p}$.

It is a consequence of the decomposition theory for D_α and $A^{2,1-2\alpha}$ (see Remark 1.2 and also [R, Theorem 2.10]) that, if $\{z_j\}$ is a d-lattice in $\mathbf{D}$ with d (>0) sufficiently small, then both

$$\left\{\frac{(1-|z_j|^2)^{3/2-\alpha}}{(1-\bar{z}_j z)^{2-2\alpha}}\right\} \quad \text{and} \quad \left\{\frac{(1-|z_j|^2)^{3/2-\alpha}}{(1-\bar{z}_j z)^{3-2\alpha}}\right\}$$

are WO sequences in D_α and $A^{2,1-2\alpha}$, respectively. Hence

$$\|f\|^p_{X^p_\alpha} \asymp \sum |\lambda_j|^p \asymp \|f\|^p_{A^{p,(2-2\alpha)p-2}}.$$

This implies $A^{p,(2-2\alpha)p-2} \subseteq X^p_\alpha$. ∎

PROPOSITION 7. *For $\alpha < \frac{1}{2}$, $p \geqslant 1$, and $(1-\alpha)p > 1$, we have*

$$X^p_\alpha = A^{p,(2-2\alpha)p-2}.$$

Proof. Let $1/p + 1/q = 1$, then $(1-\alpha)p > 1$ is equivalent to $q\alpha < 1$. By a theorem in [P, Theorem 1], for $\alpha < \frac{1}{2}$ and $q\alpha < 1$, we have that $\mathscr{H}^{(\alpha)}_b$ is in S_q if and only if b' is in $A^{q,q-2}$. Hence by Proposition 5 and Theorem 3 we get $X^p_\alpha = (A^{q,q-2})^*$. A standard argument shows that $(A^{q,q-2})^* = A^{p,(2-2\alpha)p-2}$ under the pairing $\langle f, g\rangle_{L^2(dA_\alpha)}$. The desired result then follows. ∎

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